20201013 & 16

Sequences / Cauchy sequences / limits [notes]

$\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = \{x \in \mathbb{R}, x > 0\}$

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$

$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$

$\mathbb{Q} = \{p/q,\ p, q \in \mathbb{Z},\ q \neq 0\}$

$|a \cdot b| = |a| \cdot |b|$

$\left|a/b\right| = \dfrac{|a|}{|b|} \quad (b \neq 0)$

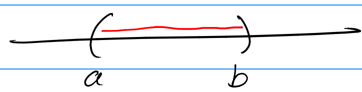
$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$

$|a+b| \leq |a| + |b|$ (The Triangle Ineq'lty)

$a < b$ $(a, b \in \mathbb{R})$

"open"

$(a, b) = \{x \in \mathbb{R},\ a < x < b\}$



$a \leq b$

"closed"

$[a, b] = \{x \in \mathbb{R},\ a \leq x \leq b\}$

* Subsets of the real line

Upper and lower bounds (supremum/infinimum)

$A \subseteq \mathbb{R}$ (A not empty)

$x \in \mathbb{R}$ is an upper bound of $A \Leftrightarrow \forall y \in A,\ y \leq x$.

$x' \in \mathbb{R}$ " " lower bound " " $\Leftrightarrow \forall y \in A,\ x' \leq y$.

We say $M \in \mathbb{R}$ is the least upper bound of A if (supremum)

M is an upper bound

and for any UB x of A, $M \leq x$

$\boxed{\text{EX1}}$ Least UB of A is unique.

Also m is the greatest lower bound of A if (infimum)

m LB

for any LB x' of A, $m \geq x'$

$\boxed{\text{EX2}}$ Where, m unique

$A_1 = \{x \in \mathbb{R},\ x^2 < 2\}$ $\qquad A_2 = \{x \in \mathbb{R},\ x^2 \leq 2\}$

$A_3 = \{x \in \mathbb{Q},\ x^2 < 2\}$

$\sup A_1 = \sqrt{2}$

$\sup A_1 = \sup A_2 = \sqrt{2} = \sup A_3$ even though sup A3 isn't in A3

$A_4 = \{x \in \mathbb{Q},\ x^2 \leq 2\}$ $\qquad \sup A_4 = \sqrt{2}$ $\qquad$ (???)

Sequences

Convention in this course

A sequence $f: \mathbb{N}^+ \to \mathbb{R}$ $f(n) = a_n$ $(n \geq 1)$

n is a dummy / bound variable here.

All sequences in this course are infinite sequences:

$a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$

A sequence is non-decreasing (increasing) if $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq \ldots$

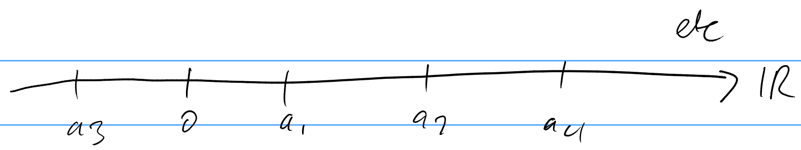
A sequence is non-increasing (decreasing) if $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq \ldots$ $\Big\} (n \geq 1)$

"strictly" if > / <

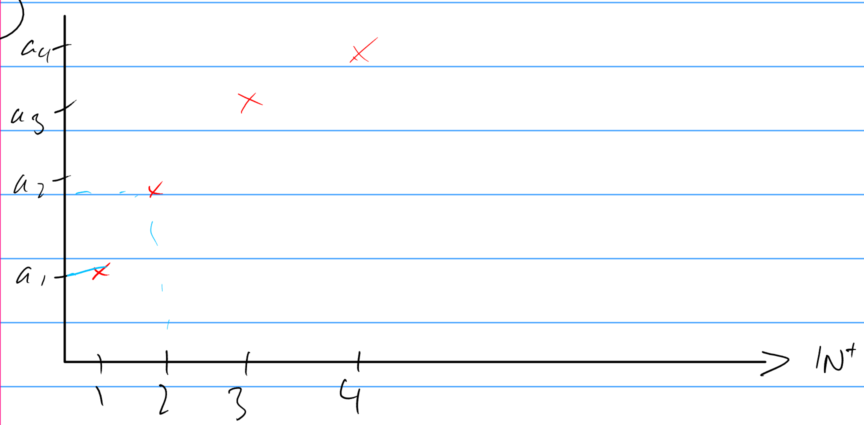
A monotonic sequence is a sequence which is *either* increasing *or* decreasing.

Graphically displaying a_n $(n \geq 1)$

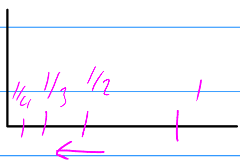
I



II



$a_n = \frac{1}{n}$?



$1 > 1/2 > 1/3 > 1/4 \ldots \Rightarrow$ sequence is *strictly* decreasing

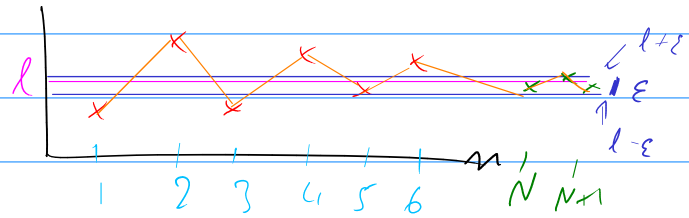
convergent

$a_n = (-1)^n$

$-1, +1, -1, +1, -1, \ldots$ sequence is alternating does not converge

Def.

a_n $(n \geq 1)$ is convergent to a limit $\ell \in \mathbb{R}$ if...



...for every a_n

where $n \geq N$

$|a_n - \ell| < \varepsilon$

where you can choose N for all values of epsilon (I think) and have this hold

Ex. $\frac{1}{n} \to 0$ as $n \to \infty$

Proof. Let $\varepsilon > 0$. $|a_n - 0| = \frac{1}{n} < \varepsilon$

$\iff n > \frac{1}{\varepsilon}$? ? ?

Take $N = \lceil \varepsilon \rceil$

$\mathbb{R} \cup \{-\infty, \infty\}$

(i.e. convergence to +/-inf allowed)

$a_n \to \infty$ as $x \to \infty$ in the extended real line $\forall k > 0$,

if for all K > 0 there exists an N so that $\forall n > N$, $a_n > K$

$a_n = n$ obviously goes to infinity as $n \to \infty$

Ex $a_n = \frac{1}{n^2+1} \to 0$ $\quad \left|\frac{1}{n^2+1} - 0\right| < \varepsilon$

Let $\varepsilon > 0$

$$\frac{1}{n^2+1} < \varepsilon$$

$$\Rightarrow n^2+1 > \frac{1}{\varepsilon}$$

$$\Rightarrow n^2 > \frac{1}{\varepsilon} - 1$$

if $n^2 > \frac{1}{\varepsilon}$, $\quad n^2 > \frac{1}{\varepsilon} - 1$

choose $N = \left\lceil \sqrt{1/\varepsilon} \right\rceil$

Useful properties:

Given
$a_n \to a$
$b_n \to b$

$\Rightarrow$ I $\alpha a_n + \beta b_n \to \alpha a + \beta b$. $(\forall \alpha, \beta \in \mathbb{R})$

II $a_n \times b_n \to ab$

III $\frac{a_n}{b_n} \to a/b$ $\quad a_n \neq 0$, $b_n \neq 0$ $\quad \forall n \geq 1$

e.g.

$$a_n = \frac{7n^2 - 5n + 4}{3n^2 + 11n - 6} = \frac{7 - \frac{5}{n} + \frac{4}{n^2}}{3 + \frac{11}{n} - \frac{6}{n^2}} \to \frac{7}{3}$$

($\frac{5}{n} \to 0$, $\frac{4}{n^2} \to 0$, $\frac{11}{n} \to 0$, $\frac{6}{n^2} \to 0$)

Tests for convergence

Sandwich theorem:

If $\begin{cases} a_n \to a \\ b_n \to a \end{cases}$ & $a_n \leq c_n \leq b_n$ $\quad \forall n \geq 1$

Then $\Rightarrow$ $\boxed{c_n \to a}$.

Proof Let $\varepsilon > 0$ be given.

Have to find an N so that $\forall n > N$ $|c_n - a| < \varepsilon$.

Using the same ε:

there exists an N_1 s.t. $\forall n > N_1$

$|a - a_n| < \varepsilon$. $\Rightarrow$ $a - \varepsilon < a_n$

similarly, there exists N_2 s.t. $\forall n > N_2$, (taking one side of the modulo)

$|a - b_n| < \varepsilon$. $\Rightarrow$ $b_n < a + \varepsilon$

$c_n \geq a_n$ & $a_n > a - \varepsilon$

$\to c_n \geq a - \varepsilon$

$b_n \geq c_n$ & $b_n < a + \varepsilon$

$\to a + \varepsilon \geq c_n$

$\to a + \varepsilon \geq c_n \geq a - \varepsilon$

$\Rightarrow |c_n - a| \leq \varepsilon.$

Using this result:

$a_n: \frac{1}{n} \to 0$ as $n \to \infty$

$\to c_n: \frac{1}{n^5} \to 0$ as $n \to \infty$

& $b_n = 0$ $\forall n$ $(\Rightarrow b_n \to 0)$.

$\frac{1}{n} \geq \frac{1}{n^5} \geq 0$

$\to a_n \geq c_n \geq b_n$ $\Rightarrow$ $\boxed{c_n \to 0}$

$c_n = \frac{\sin \frac{1}{n}}{n}$

$-\frac{1}{n} \leq \frac{\sin \frac{1}{n}}{n} \leq \frac{1}{n}$

$a_n = -1/n \to 0$

$b_n = 1/n \to 0$

since $\sin x \in [-1, 1]$ $\forall x$

$\to \boxed{c_n \to 0}$

Ratio test for convergence

* Suppose that $\left|\frac{a_n}{a_{n+1}}\right| \leq r < 1$ for $n \geq N_0$

for specified r, N_0

then that implies that $a_n \to 0$.

$|a_{N_0+2}| \leq |a_{N_0+1}| r \leq |a_{N_0}| r^2$

$\to |a_{N_0+r}| \leq |a_{N_0}| r^2$

** Proof

Let $\varepsilon > 0$ spec. need to find N so that $\forall m > N$,

$$|a_m| < \varepsilon.$$

$$\left(\frac{\text{Assume}}{|a_m| < \varepsilon}\right) \quad |a_{N_0+m}| \leq |a_{N_0}| \, r^n \qquad ???$$

Limit ratio test:

If $\left|\frac{a_{n+1}}{a_n}\right| \to S$, as $n \to \infty$

(for some $S < 1$)

$$|a_n| \to 0.$$

Proof

Let $\varepsilon = \frac{1-S}{2}$ (we can choose this, since we're already given a limit to S)

then by definition $\exists N$ s.t.

$n > N$

$$\left| \left|\frac{a_{n+1}}{a_n}\right| - r \right| < \frac{1-r}{2}$$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| < \frac{1-r}{2} < 1$$

$\Rightarrow$ use the previous Ratio Test, and the result follows.

$a_n = \frac{1}{n!}$ $\left(0 < a_n < \frac{1}{n}, \text{ so } a_n \to 0\right)$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{n+1} \to 0$$

$\Rightarrow a_n \to 0$

Subsequences

$f: \mathbb{N}^+ \to \mathbb{R}$ $\quad f(n) = a_n$

Given $a_1, a_2, a_3 \dots$

we're selecting certain elements from the sequence, creating a subsequence:

$a_{n_1}, a_{n_2}, a_{n_3}, \dots$ $\quad g: A \to \mathbb{R}$ $\quad A \subseteq \mathbb{N}^+$, A infinite.

So, a subsequence restricts the elements to a subset. "$(a_{n_i})_{i \geq 1}$"

$n_i \geq i \quad \forall i$

$a_n = (-1)^n \quad -1, 1, -1, 1, \dots$

Define

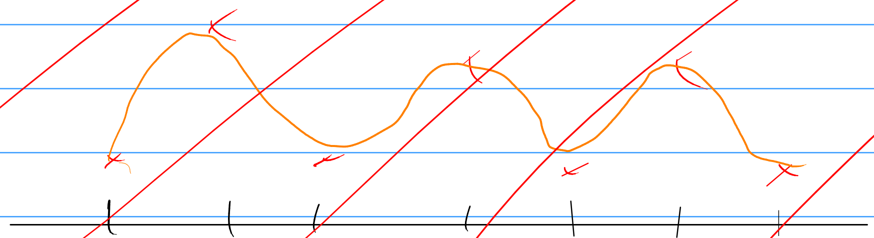
$a_{n_i} = 2i+1 \Rightarrow n_1 = 1, n_2 = 3, n_3 = 5$ etc

$\Rightarrow a_{n_i}: -1, -1, -1, \dots$

[SEE PROOF ON NEXT PAGE]

Why are these subsequences useful? Well, there's an interesting result:

if a_n is any sequence, then a_n has a monotonic subsequence. (a_{n_i})



Define a peak of the sequence a_m:

$\forall n \geq m \quad a_m \geq a_n$

Case 1 There are infinitely many peaks.

The sequence of the peaks is (apparently) decreasing

$a_{m_1}, a_{m_2}, a_{m_3}, \dots$

Case 2: There's a finite number of peaks.

$p_{m_1}, p_{m_2}, \dots p_{m_k}.$

(new topic)

Proposition: If $a_n \to \ell$ $(\ell \in \mathbb{R})$, then for any subsequence $a_{n_i} \to \ell$,

$a_{n_i} \to \ell$ as $i \to \infty$.

Proof: (given from $a_n \to \ell$)

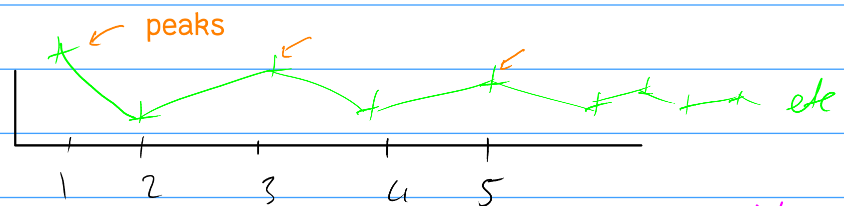
let $\varepsilon > 0 \to$ there exists an N such that $\forall n > N, |a_n - \ell| < \varepsilon$.

For $i > N$, we know already that $n_i \geq i \; \forall i \Rightarrow n_i \geq i > N$.

Hence $|a_{n_i} - \ell| < \varepsilon$. $\square$

Proposition: any sequence a_n of real numbers has a subsequence a_{n_i} such that the subsequence is either increasing or decreasing (monotonic)

Proof:



a_m is a peak of the sequence if: $a_m \geq a_n \; \forall n \geq m$.

Case 1: there exist infinitely many peaks of the sequence a_n.

$a_{n_1}, a_{n_2}, a_{n_3}, \ldots \leftarrow$ peaks

Then this sequence is monotonically decreasing (by definition).

Case 2: there are a finite number of peaks (incl. none at all)

Take the last peak of the finite set of peaks; call it a_N

Then specify $n_1 = N+1$ in a_{n_i}

$a_{n_1} = a_{N+1}$ is not a peak, so there exists

n_2 s.t. $a_{n_1} < a_{n_2}$ $(n_2 > n_1)$

(by def. of not being a peak)

$\to$ there also exists an $a_{n_3} > a_{n_2}, \ldots$

$\to$ continue iteratively to get a monotonically increasing seq.

Completeness axiom of real numbers

If $A \subseteq \mathbb{R}$ is a non empty, upper bounded subset of $\mathbb{R}$,

then $\sup A$ exists.

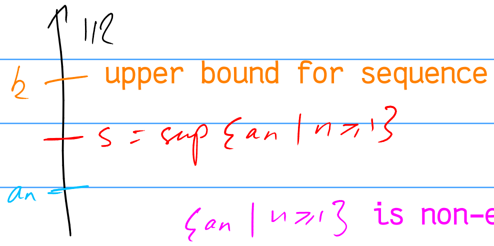
(Similarly, if the set is lower bounded, the infimum exists)

Proposition: If a_n is an increasing sequence bounded above, then

$$a_n \to l \quad \text{as } n \to \infty. \qquad (l \in \mathbb{R})$$

(this also holds for seqs. bounded below → the infimum)

Proof:



$\{a_n \mid n \geq 1\}$ is non-empty and bounded above

→ by the Completeness Axiom, $s = \sup\{a_n \mid n \geq 1\}$ exists.

To prove $a_n \to s$ let the given $\varepsilon > 0$.

Claim there exists N such that $s - \varepsilon < a_N \leq s$

If this doesn't hold, then $\forall n$, $s - \varepsilon \geq a_n$

→ $s - \varepsilon$ is an upper bound by the def. of an upper bound

→ it's the supremum of the set →←

Thus the claim holds, and the sequence converges on the supremum.

Cauchy sequences

Def A sequence (a_n) is a Cauchy sequence if $\forall \varepsilon > 0$ there exists N such that $\forall n,m > N$ $|a_n - a_m| < \varepsilon$.

Prop. If $a_n \to l \in \mathbb{R}$ as $n \to \infty$, then a_n is a Cauchy sequence.

Proof Let $\varepsilon > 0$ be given.

There exists an N such that $\forall n > N$ $|a_n - l| < \frac{\varepsilon}{2}$ (valid because of the def. of a limit)

Then $|a_n - a_m| \leq |a_n - l| + |l - a_m|$ by the Triangle Inequality (see p1)

$< \varepsilon/2 + \varepsilon/2$ $\quad \forall n,m > N$

$\to |a_n - a_m| < \varepsilon$ $\square$

Theorem Every Cauchy sequence a_n is convergent to some $l \in \mathbb{R}$ (the converse of the above).

Proof (1) If a sequence is Cauchy, it is bounded both above and below:

Let's specify our $\varepsilon = 1$ in the Cauchy seq. definition

$\to$ there exists N s.t. $|a_n - a_m| < 1$ $\forall n,m > N$

let's say $a_m = a_{(N+1)}$

$\Rightarrow |a_n - a_{N+1}| < 1.$

$|a_n| = |a_n - a_{N+1} + a_{N+1}|$

$\Rightarrow |a_n| \leq |a_n - a_{N+1}| + |a_{N+1}|$ (triangle inequality)

$< 1 + |a_{N+1}|$

Therefore $|a_n| \leq \max\{|a_1|, |a_2|, \ldots, |a_N|, |a_{N+1}| + 1\}$ $\forall n \geq 1$

(2) Any subsequence is bounded (you can't have an unbounded subseq. of a bounded seq.)

(3) In particular we know there exists a monotonic, bounded subsequence a_{n_i}.

(4) Using the previous results, the previous sequence converges on either its supremum or infimum (depending on which way it's bounded)

(5) We now propose and prove that, if $a_{n_i} \to l \in \mathbb{R}$, $a_n \to l$ as $n \to \infty$:

(remember, the sequence is Cauchy; this isn't generically true)

Let $\varepsilon > 0$ $\to \exists N_1$ s.t. $\forall n,m > N_1$, $|a_n - a_m| < \varepsilon/2$.

$\exists N_2$ s.t. $\forall i > N_2$, $|a_{n_i} - l| < \varepsilon/2$.

$|a_n - l| \leq |a_n - a_{n_i}| + |a_{n_i} - l|$ (triangle inequality again)

let $N = \max\{N_1, N_2\}$

$\forall n, i > N$, $|a_n - a_{n_i}| + |a_{n_i} - l| < \varepsilon$

$\boxed{\Rightarrow |a_n - l| < \varepsilon.}$ $\square$

Def the set A is "complete" if any Cauchy sequence in A has a limit in A $(A \subseteq \mathbb{R})$

Let $A = \mathbb{R}$ $\Rightarrow$ $\mathbb{R}$ is complete (by the proofs on the last page)