Functions and series

Sandwich theorem examples

Ex Show that, for p > 0, $p^{1/n} \to 1$ as $n \to \infty$

Assume that $p > 1$ Let $x_n = p^{1/n} - 1 \quad (>0)$

$\Rightarrow p^{1/n} = 1 + x_n$

$\Rightarrow p = (1 + x_n)^n$

n is a positive integer, so using binomial theorem:

$p > 1 + n x_n$

$$\Rightarrow \underset{\to 0}{\underset{\uparrow}{0}} < x_n < \underset{\to 0}{\underset{\uparrow}{\frac{p-1}{n}}}$$

$\Rightarrow x_n \to 0$ (Sandwich th'm)

$\Rightarrow \underline{\underline{p^{1/n} \to 1}}$

Ex Show that $n^{\frac{1}{n}} \to 1$ as $n \to \infty$.

for $n \geq 2$ $\quad n^{\frac{1}{n}} > 1$

Let $x_n = n^{\frac{1}{n}} - 1 \quad (n \geq 2$

$\longrightarrow x_n > 0$

$n^{\frac{1}{n}} = (1 + x_n)$

$\Rightarrow n = (1 + x_n)^n > \frac{n(n-1)}{2} x_n^2$ (binomial theorem)

$0 < x_n^2 < \frac{2}{n-1} \quad (n \geq 2)$

$$\Rightarrow \underset{\to 0}{\underset{\uparrow}{0}} < x_n < \underset{\to 0}{\underset{\uparrow}{\frac{\sqrt{2}}{\sqrt{n-1}}}}$$

$\Rightarrow x_n \to 0$

$\Rightarrow \underline{\underline{n^{\frac{1}{n}} \to 1}}$

Functions

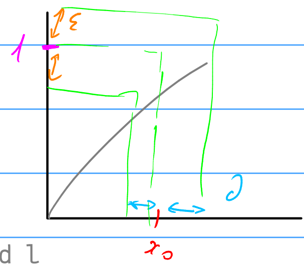
$f: A \to \mathbb{R}$ $\qquad$ $A = \mathbb{R}$ or $[a,b]$, (a,b), $[a,b)$, $(a,b]$, (a,∞) etc.

If $x_0 \in A$, $\lim_{x \to x_0} f(x) = l$ $\quad (l \in \mathbb{R})$

if $\forall \varepsilon > 0$, exists $\delta > 0$

so that $\forall x$ $|x - x_0| < \delta$,

$|f(x) - l| < \varepsilon$



(so you're given an epsilon around l
and you need to define a delta around x0
to constrain x so the f(x) thing works).

Ex $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

Claim $\lim_{x \to 0} f(x) = 0.$

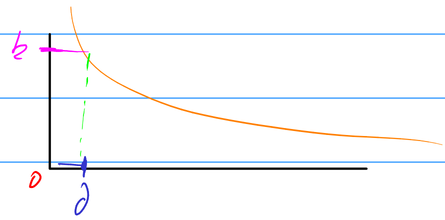
Proof Let $\varepsilon > 0.$ $\quad |x \sin \frac{1}{x} - 0| < \varepsilon$

$|x \sin \frac{1}{x} - 0| \le |x| < \varepsilon \Rightarrow$ set $\delta = \varepsilon$

$|x - 0| = |x| < \varepsilon \Rightarrow |x \sin \frac{1}{x} - 0| \le |x| < \varepsilon.$

Ex $g(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ $\qquad \lim_{x \to 0} g(x).$ does not exist

Ex $h(x) = \frac{1}{x}$ $\qquad h: (0, \infty) \to \mathbb{R}$



Claim $\lim_{x \to 0} h(x) = \infty.$

Proof $\forall k > 0$ there must exist $\delta > 0$ so that

$\forall x \ |x - 0| < \delta, \ h(x) > k.$

$\Rightarrow \frac{1}{x} > k$

$\Rightarrow x < \frac{1}{k}$

so choose $\delta = \frac{1}{k}$

!? If $\lim_{x\to x_0} f(x) = l_1$ & $\lim_{x\to x_0} g(x) = l_2$.

$$\lim_{x\to x_0} af(x)+bg(x) = al_1+bl_2.$$

Continuous functions

Def $f:(a,b)\to\mathbb{R}$ is continuous at $x_0 \in (a,b)$

if $\lim_{x\to x_0} f(x)$ exists ①   example: $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

& $\lim_{x\to x_0} f(x) = f(x_0)$ ②.

Composition

① If f & g are continuous at x_0, then af + bg is continuous at x_0.

$f: A\to\mathbb{R}$ $g: B\to\mathbb{R}$

& $f(x)\subseteq B \Rightarrow (g\circ f)$ is well defined on A

② If f is continuous at x_0, and g is continuous at $f(x_0)$,

then g · f is continuous at x_0.

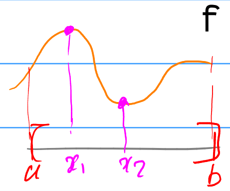
(not examinable!)

Three basic properties of continuous functions

$f:(a,b)\to\mathbb{R}$ is cts. if it is cts. at $x_0 \in (a,b)$ $\forall x_0$.

① If $f:[a,b]\to\mathbb{R}$ is continuous, then

f is bounded & attains its min & max.
(x_1) (x_2)

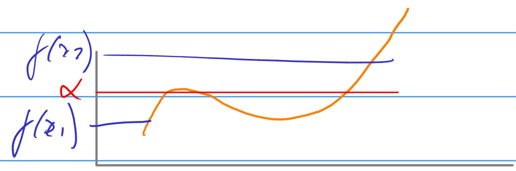


The interval does have to be closed (i.e. [a, b] not (a, b)) --
consider cases like $f(x) = \frac{1}{x}$ $x\in(0,\infty]$

② If $f:[a,b]\to\mathbb{R}$ is continuous, then

$\forall \alpha\in[a,b]$ $f(a) < \alpha < f(b) \Rightarrow \exists x\in(a,b)$

s.t. $f(x)=\alpha$.



③ If $f:[a,b]\to\mathbb{R}$ is continuous, then it is uniformly continuous.

$\forall \varepsilon>0\ \exists\delta>0$ s.t. $\forall x\ |x-x_0|<\delta$, $|f(x)-f(x_0)|<\varepsilon$ $\forall x_0$

(this being at the end is important)

"uniformly continuous" means that the value of δ

has no dependency on the value of x_0.

$f(x) = x^2 \qquad f : [a,b] \to \mathbb{R} \qquad 0 \le a < b$

THIS BIT IS DODGY AND PROBABLY INCORRECT

Let $\varepsilon > 0$ Then $|x_0^2 - x^2| = |x_0 - x||x_0 + x|$

$\le |x_0 - x| \, 2b < \varepsilon$

take $\delta = \varepsilon / 2b$.

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Series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$$

$$G(x) = x + x^2 + x^3 + x^4 + \dots + x^n \qquad |x| < 1$$

$$= x(1 + x + x^2 + x^3 + \dots)$$

$$= x(1 + G(x))$$

$$\Rightarrow G(x)(1-x) = x$$

$$\Rightarrow G(x) = \frac{x}{1-x}.$$ (standard geometric progression)

$$G_n(x) = x + x^2 + \dots + x^n$$

$$G_n(x) = \frac{x(1-x^n)}{1-x} = \frac{x}{1-x} - \frac{x^{n+1}}{1-x} \to \frac{x}{1-x}$$

While G(x) and Gn(x) are familiar from A-level, here we've found G(x) by finding $\lim_{n\to\infty} G_n(x) = l \in \mathbb{R}$.

Generally

- $S = a_1 + a_2 + a_3 + \dots + a_n + \dots$
- $S_n = a_1 + a_2 + \dots + a_n$ ("partial sum")

If $S_n \to S \in \mathbb{R}$ then we say the series converges to S. Otherwise, it diverges

Harmonic series $H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges

$$1 + \underbrace{\frac{1}{2}}_{=\frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{4} + \frac{1}{4}\ >\ 2\times\frac{1}{4}\ \geq\ \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq\frac{1}{8},\ \geq\frac{1}{8},\ \geq\frac{1}{8},\ \frac{1}{8}\ >\ 4\times\frac{1}{8}\ \geq\ \frac{1}{2}} + \underbrace{\dots + \frac{1}{16}}_{>\frac{1}{2}} \quad \text{etc.}$$

$$S_{2^n} \geq 1 + \frac{n}{2}$$

$$\Rightarrow S_{2^n} \to \infty$$

WARNING: This is necessary,
but NOT sufficient (see H)

$\Rightarrow S_n$ diverges

Prop. If $S_n \to l \in \mathbb{R}$, then $a_n \to 0$ as $n \to \infty$.

Proof $S_n \to l \in \mathbb{R} \Rightarrow S_n$ is a Cauchy sequence.

$$\forall \varepsilon > 0\ \exists N \text{ s.t. } |S_n - S_m| < \varepsilon \quad \forall m, n > N.$$

Let $\varepsilon > 0$ be given for proving $a_n \to 0$.

Thus by the Cauchy property $\exists N \mid |S_m - S_n| < \varepsilon \quad \forall n, m > N$

Let $n, m \geq N+1$, $m = n+1$.

$$\Rightarrow |S_{n+1} - S_n| = |a_{n+1}| < \varepsilon$$

$$\Rightarrow |a_{n+1} - 0| < \varepsilon \quad \forall n \geq N+1$$

$$\Rightarrow |a_n - 0| < \varepsilon \quad \forall n \geq N+2.$$

$$\Rightarrow \underline{a_n \to 0}$$

$S = \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2}$ converges:

first, check that $\frac{1}{n^2} \to 0$ as $n \to \infty$ ✓

$a_n = \frac{1}{n^2}$

$$\frac{1}{n(n+1)} < \frac{1}{n^2} < \frac{1}{n(n-1)} \quad n \geq 2$$

$$b_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S'_n = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \dots + \left(\cancel{\tfrac{1}{n}} - \tfrac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\sum_{n \geq i \geq 2} \frac{1}{i(i+1)} < \sum_{n \geq i \geq 2} \frac{1}{i^2} < \sum_{n \geq i \geq 2} \frac{1}{i(i-1)}$$

$$\Rightarrow 1 - \frac{1}{n+1} - \frac{1}{2} < \sum_{n \geq i \geq 2} \frac{1}{i^2} < 1 - \frac{1}{n}$$ (using the S' thing above)

$$\Rightarrow 3/2 - \frac{1}{n+1} < S_n < 2 - \frac{1}{n}$$ (changing i⩾1)

We can't do the sandwich theorem because they don't converge to the same thing :(

S_n is an increasing sequence: $S_{n+1} > S_n$

$\ell\, S_n < 2$ $\quad \Rightarrow S_n \to \ell, \; 3/2 \leq \ell \leq 2$